DIRICHLET PROBLEM FOR A TOROIDAL SEGMENT

(ZADACHA DIRIKHLE DLIA TOROIDAL'NOGO SEGMENTA)

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We present the exact solution of the Dirichlet problem for a region bounded by the surfaces of a torus, two intersecting spheres and two semi-planes passing through the axis of symmetry. In the course of our proof we found it necessary to derive a special transformation formula for an integral transformation in spherical functions with complex indices, representing a generalization of the Mehler-Fock transformation. Possible application of the obtained results to some problems of mathematical physics and of the theory of elasticity, are indicated.

1. Introducing the toroidal coordinates connected with the Cartesian system by

$$x = \frac{a \sinh\alpha \cos\varphi}{\cosh\alpha + \cos\beta}, \qquad y = \frac{a \sinh\alpha \sin\varphi}{\cosh\alpha + \cos\beta}, \qquad z = \frac{a \sin\beta}{\cosh\alpha + \cos\beta} \tag{1.1}$$

we obtain the following equations for the boundary surfaces

$$\alpha = \alpha_0, \qquad \beta = \beta_1, \qquad \beta = \beta_2, \qquad \varphi = 0, \qquad \varphi = \gamma$$

(see Fig. 1 showing the meridional cross section).

We shall seek the solution $u(\alpha, \beta, \varphi)$ of the Laplace's equation in the region $\alpha_0 < \alpha < \infty$, $\beta_1 < \beta < \beta_2$, $0 < \varphi < \gamma$, satisfying the following boundary conditions:

$$u(\alpha, \beta, 0) = u(\alpha, \beta, \gamma) = u(\alpha_0, \beta, \varphi) = 0$$

$$u(\infty, \beta, \varphi) < \infty \tag{1.2}$$

$$u(\alpha, \beta_k, \varphi) = f_k(\alpha, \varphi) \qquad (k = 1,2) \tag{1.3}$$

where $f_k(\alpha, \varphi)$ are given functions. If the solution of the stated problem is sought in the form of a Fourier series

$$u = \sum_{m=1}^{\infty} u_m(\alpha, \beta) \sin \mu_m \varphi, \qquad \mu_m = \frac{m\pi}{\gamma} \tag{1.4}$$

then for u_m we have (1.5)

$$\frac{\partial}{\partial\alpha}\left(\frac{r}{a}\frac{\partial u_m}{\partial\alpha}\right) + \frac{\partial}{\partial\beta}\left(\frac{r}{a}\frac{\partial u_m}{\partial\beta}\right) - \frac{\mu_m^2 u_m}{\sinh\alpha\,(\cosh\alpha + \cos\beta)} = 0$$

Substitution [1]

$$u_m = \sqrt{\cosh\alpha + \cos\beta}\, F(\alpha)\, \Phi(\beta) \tag{1.6}$$

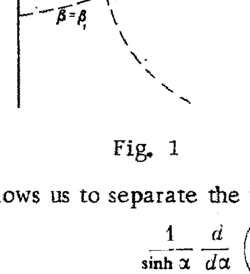

Fig. 1

allows us to separate the variables and obtain

$$\frac{1}{\sinh\alpha}\frac{d}{d\alpha}\left(\sinh\alpha \frac{dF}{d\alpha}\right) - \left[\nu(\nu+1) + \frac{\mu_m^2}{\sinh^2\alpha}\right]F = 0 \tag{1.7}$$

$$\frac{d^2\Phi}{d\beta^2}+\left(\nu+\frac{1}{2}\right)^2\Phi=0 \tag{1.8}$$

Changing in (1. 7) the variable $x=\cosh\alpha$, assuming that $F(\alpha)=y(x)$ and taking the boundary conditions $F(\alpha_0)=0$ and $F(\infty)<\infty$ into account, we arrive at the following singular boundary value problem :

$$[(x^2-1)y']'+\left(\lambda-\frac{\mu^2}{x^2-1}\right)y=0,\quad x_0<x<\infty,\quad y(x_0)=0,\qquad y(\infty)<\infty \tag{1.9}$$

where

$$x_0=\cosh\alpha_0,\qquad \lambda=-\nu(\nu+1),\qquad \mu\equiv\mu_m \tag{1.10}$$

We shall show that this problem has a continuous spectrum of eigenvalues.

Considering first the region $\operatorname{Re}\nu>0$ we find, that the Legendre functions $y=Q_\nu^\mu(x)$ will be eigenfunctions, while the eigenvalues will be given by

$$Q_\nu^\mu(x_0)=0 \tag{1.11}$$

Expansion of $Q_\nu^\mu(x)$ into a hypergeometric series shows at once that the equality (1. 11) cannot be fulfilled for real values of $\nu>-1$. Let us assume that the eigenvalues are complex and let us construct, with the aid of (1. 9), the following integral relation :

$$(\lambda-\bar\lambda)\int_{x_0}^{\infty}y\bar y\,dx=\lim_{x\to\infty}[(x^2-1)(\bar y y'-\bar y y')] \tag{1.12}$$

Here y and $\bar y$ are complex conjugates. Going to the limit with $\operatorname{Re}\nu>-\frac{1}{2}$ we obtain the identity $y\equiv 0$ which contradicts the previous assumption. Thus, for $\operatorname{Re}\nu>0$ Equation (1. 11) has no roots, i. e. no eigenvalues.

It remains to consider the region (*)

$$-1<\operatorname{Re}\nu\leqslant 0 \tag{1.13}$$

where eigenfunctions exist, which can be expressed in terms of Legendre functions of first and second kind in the following manner :

$$y=C\,[Q_\nu^\mu(x_0)P_\nu^\mu(x)-P_\nu^\mu(x_0)Q_\nu^\mu(x)] \tag{1.14}$$

Thus, the boundary value problem (1. 9) has a continuous spectrum of eigenvalues contained within the strip (1. 13).

It is convenient to put

$$\nu=-\frac{1}{2}+i\tau,\qquad C=\frac{e^{-i\mu\pi}}{\cos(\mu-i\tau)\pi} \tag{1.15}$$

since, on this substitution, both the eigenfunctions and the eigenvalues ($\lambda=\tau^2+\frac{1}{4}$) become real.

Solving next (1. 8), we arrive at the following expression for the functions $u_m(\alpha,\beta)$ to be determined

$$u_m=\sqrt{\cosh\alpha+\cos\beta}\int_0^\infty[A_2(\tau)\sinh\tau(\beta-\beta_1)+A_1(\tau)\sinh\tau(\beta_2-\beta)]\frac{y(x,\tau)\,d\tau}{\sinh\tau(\beta_2-\beta_1)} \tag{1.16}$$

$$y(x,\tau)=\frac{e^{-i\mu\pi}}{\cos(\mu-i\tau)\pi}[Q_{-1/2+i\tau}^{\mu}(x_0)P_{-1/2+i\tau}^{\mu}(x)-P_{-1/2+i\tau}^{\mu}(x_0)Q_{-1/2+i\tau}^{\mu}(x)] \tag{1.17}$$

It remains to satisfy the nonhomogeneous condition (1. 3) expressed as

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*) The region $\operatorname{Re}\nu-1$ need not be considered separately, since the eigenvalues do not change when ν is replaced by $-\nu-1$.

$$F_k(x) = \int_0^\infty A_k(\tau)\, y(x, \tau)\, d\tau \qquad (k = 1, 2) \tag{1.18}$$

the left-hand sides of which, in terms of given functions $f_k(\alpha, \varphi)$, are

$$F_k(x) = \frac{2}{\gamma} (\cosh\alpha + \cos\beta_k)^{-1/2} \int_0^\gamma f_k(\alpha, \varphi) \sin\mu\varphi\, d\varphi \tag{1.19}$$

In this manner the problem under consideration leads to the necessity of expanding the given function $f(x)$ into an integral in eigenfunctions of the boundary value problem (1.9)

$$f(x) = \int_0^\infty F(\tau)\, y(x, \tau)\, d\tau \qquad (1 < x_0 < x < \infty) \tag{1.20}$$

2. Relation allowing the determination of $F(\tau)$ from the expansion (1.20), has the form

$$F(\tau) = \frac{\pi\tau \sinh \pi\tau}{|\Gamma(1/2 + i\tau + \mu|^2 |Q^{\mu}_{-1/2+i\tau}(x_0)|^2} \int_{x_0}^\infty f(x)\, y(x, \tau)\, dx \tag{2.1}$$

and can formally be obtained by a method similar to that of Titchmarsh [2]. Consider the solution of the following partial differential equation

$$\frac{\partial}{\partial x}\left[(x^2 - 1)\frac{\partial u}{\partial x}\right] - \frac{\mu^2}{x^2 - 1} u = \frac{\partial u}{\partial t} \tag{2.2}$$

with the conditions

$$u(x_0, t) = 0, \qquad u(\infty, t) < \infty, \qquad u(x, 0) = f(x) \tag{2.3}$$

Applying Laplace transforms, we obtain the solution of this problem in the form

$$u(x, t) = e^{-i\mu\pi} \int_{x_0}^\infty f(\xi)\, d\xi \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \frac{\Gamma(\nu - \mu + 1)}{\Gamma(\nu + \mu + 1)} \frac{G(x, \xi, p)}{Q^{\mu}_{\nu}(x_0)} e^{pt} dp \tag{2.4}$$

where

$$G(x, \xi, p) = \begin{cases} U_1(x)\, U_2(\xi) & (x \leqslant \xi) \\ U_1(\xi)\, U_2(x) & (x \geqslant \xi) \end{cases} \tag{2.5}$$

$$U_1 = Q^{\mu}_{\nu}(x_0)\, P^{\mu}_{\nu}(x) - P^{\mu}_{\nu}(x_0)\, Q^{\mu}_{\nu}(x), \qquad U_2 = Q^{\mu}_{\nu}(x) \tag{2.6}$$

$$\nu = -1/2 + \sqrt{p + 1/4}, \qquad \operatorname{Re}\sqrt{p + 1/4} > 0 \tag{2.7}$$

Transforming the complex integral appearing in (2.4) into the sum of integrals along the sides of a cut made along the segment $(-\infty, -\frac{1}{4})$ of the real axis of the plane of the complex variable p, changing the order of integration in (2.4) and putting $t = 0$, we obtain

$$f(x) = \pi \int_0^\infty \frac{\tau \sinh \pi\tau\, y(x, \tau)\, d\tau}{|\Gamma(1/2 + i\tau + \mu)|^2 |Q^{\mu}_{-1/2+i\tau}(x_0)|^2} \int_{x_0}^\infty f(\xi)\, y(\xi, \tau)\, d\xi \tag{2.8}$$

which, equated to (1.20), yields (2.1).

Let us introduce the integral transformation

$$F(\tau) = \int_{x_0}^\infty f(x)\, y(x, \tau)\, dx \tag{2.9}$$

of $f(x)$ in terms of eigenfunctions $y(x, \tau)$; by (2.8), we obtain the following transformation formula:

$$f(x) = \frac{\pi\tau \sinh \pi\tau}{|\Gamma(1/2 + i\tau + \mu)|^2 |Q^{\mu}_{-1/2+i\tau}(x_0)|^2} \int_0^\infty F(\tau)\, y(x, \tau)\, d\tau \tag{2.10}$$

which is, apparently, new.

Next we shall show some particular cases of the obtained expansion.

When $x_0 = 1$, the toroidal segment under consideration becomes a sector $(0 < \varphi < \gamma)$ of a solid lens $(0 \leq \alpha \leq \infty,\ \beta_1 < \beta < \beta_2)$. Corresponding boundary value problem transforms (2. 8) into

$$f(x) = \frac{1}{\pi} \int_0^\infty \tau \sinh \pi\tau \left| \Gamma\left(\frac{1}{2} + i\tau - \mu\right)\right|^2 P_{-1/2+i\tau}^{\mu}(x)\, d\tau \int_1^\infty f(\xi)\, P_{-1/2+i\tau}^{\mu}(\xi)\, d\xi \tag{2.11}$$

Another variant is obtained when $\mu = m$ $(m = 0, 1, 2, \ldots)$, in which case (2. 8) yields

$$f(x) = (-1)^m \int_0^\infty \frac{\Gamma(1/2 + i\tau - m)\,\tau \tanh \pi\tau\, y_m(x, \tau)}{\Gamma(1/2 + i\tau + m)\, |Q_{-1/2+i\tau}^{m}(x_0)|^2}\, d\tau \int_{x_0}^\infty f(\xi)\, y_m(\xi, \tau)\, d\xi \tag{2.12}$$

$$y_m(x, \tau) = Q_{-1/2+i\tau}^{m}(x_0)\, P_{-1/2+i\tau}^{m}(x) - P_{-1/2+i\tau}^{m}(x_0)\, Q_{-1/2+i\tau}^{m}(x) \tag{2.13}$$

This corresponds either to the value $\gamma = \pi$ or to the conditions of the fourth kind at the ends of the interval $0 \leq \varphi \leq 2\pi$ (a "complete" toroidal segment), and in the latter case the expansion (1. 4) is made in terms of $e^{im\varphi}$ from $-\infty$ to $+\infty$.

When both limit transitions ($x_0 = 1$, $\mu = m$) occur simultaneously, Formulas (2. 11) or (2. 12) yield the integral Mehler-Fock expansion [3 and 4] (*) in terms of associated Legendre Functions

$$f(x) = (-1)^m \int_0^\infty \frac{\Gamma(1/2 + i\tau - m)}{\Gamma(1/2 + i\tau + m)}\, \tau \tanh \pi\tau\, P_{-1/2+i\tau}^{m}(x)\, d\tau \int_1^\infty f(\xi)\, P_{-1/2+i\tau}^{m}(\xi)\, d\xi \tag{2.14}$$

3. Results obtained in Sections 1 and 2 allow us to obtain solutions for boundary value problems of electrostatics, heat conductivity, and so on, for the considered regions.

As an example, we shall show an interesting limiting case of a torus with a cut, when $\beta_2 = -\beta_1 = \pi$, and $\mu = m$.

This is a set-up obtained during the determination of electrostatic field between a toroidal envelope ($\alpha = \alpha_0$) and a plane ring ($\beta = \pm\pi$), maintained at different potentials.

Transformation formula obtained can also be used in some problems of the theory of elasticity, in particular in the problem on torsion of a "complete" toroidal segment under the action of forces applied to the spherical parts of the surface. A similar problem [5] reduces to solution of equation for the stress function Φ

$$\Delta\Phi - \frac{4}{r}\frac{\partial\Phi}{\partial r} = 0 \tag{3.1}$$

with the values of the sought function given on the surfaces $\beta + \beta_k$ and $\alpha = \alpha_0$; it can be considered that $\Phi = 0$ when $\alpha = \alpha_0$.

Putting $\Phi = r^2 W$ we obtain, for W

$$\Delta W - \frac{4}{r^2} W = 0 \tag{3.2}$$

Last equation becomes zero for $\alpha = \alpha_0$, and its particular solutions are

$$W_\tau(\alpha, \beta) = \sqrt{\cosh\alpha + \cos\beta}\, e^{\pm\beta\tau} y_2(x, \tau), \qquad x = \cosh\alpha \tag{3.3}$$

Functions y_2 are found from (2. 13) by putting $m = 2$. In this manner the solution

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*) See also N. N. Lebedev, Some integral transformations of mathematical physics. Author's doctoral dissertation. Leningrad, 1951.

of our problem can be represented by

$$\Phi = \frac{\sinh^2 \alpha}{(\cosh\alpha + \cos\beta)^{3/2}} \int_0^\infty [A_2 \sinh\tau(\beta - \beta_1) + A_1 \sinh\tau(\beta_2 - \beta)] \frac{y_2 d\tau}{\sinh\tau(\beta_2 - \beta_1)} \quad (3.4)$$

and the unknown quantities $A_k(\tau)$ ($k = 1, 2$) can be found at the given values of Φ when $\beta = \beta_k$, by means of the transformation formula (2.12), as follows

$$A_k = \frac{\tau\tanh\pi\tau}{(1/4 + \tau^2)(9/4 + \tau^2)\,|\,Q^2_{-1/2+i\tau}(x_0)\,|^2} \int_{\alpha_0}^\infty \Phi(\alpha, \beta_k)(\cosh\alpha + \cos\beta_k)^{3/2} \frac{y_2 d\alpha}{\sinh\alpha} \quad (3.5)$$

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